

Article

Mass- and Energy-Balance Relations in Lumped Parameter Hydraulic Modeling

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Abstract

Lumped parameter models are widely used as approximations of reality for system design and dynamic simulation due to their often simple and intuitive physical principles. This is particularly convenient, if the modeling is confined to one single physical domain, for instance, dynamic models of electronic circuits. Hydraulic drive technology is an interaction of physical effects from multiple domains such as mechanics, fluid dynamics and thermodynamics, which in turn makes the development of lumped parameter models more involved. In order to get simple models certain assumptions result in approximations that sometimes leave the correct physical context to some extent, which may lead to unexpected dynamic behaviour of certain models, like closed hydraulic systems. In this paper the problems of conventional hydraulic lumped parameter modeling regarding mass and energy conservation are addressed. The deficiencies of the established models from literature are investigated by simulations and analysed with an energy-based modeling framework from systems theory, which in turn enables the development of a simple strategy to adapt the original models in order to achieve a mass-conservative and energy-consistent behaviour under stated assumptions, even with existing simulation tools. Furthermore, the approximation of wave propagation effects in transmission lines by lumped parameter models is investigated by simulations.

Keywords: hydraulic; lumped parameter; dynamic modeling; passivity; mass balance; energy balance; compression ratio



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1. Introduction

Hydraulic drive systems are indispensable in industrial applications, ranging from heavy duty machinery and rolling mills to mobile equipment and aerospace actuation, owing to their superior power density, robustness, and capability to generate large forces within compact mechanical dimensions. Accurately predicting the dynamic behaviour of such systems, however, poses a significant modeling challenge, as it requires capturing the interaction of compressible fluid dynamics, mechanical inertia, and nonlinear valve or pump characteristics across a wide range of operating conditions. Lumped parameter models offer an attractive compromise between physical fidelity and computational effort, enabling system-level analysis, controller design, and simulation by ODEs and without resorting to the full complexity of distributed-parameter (PDE) or computational fluid dynamics (CFD) approaches.

The models widely used for decades in hydraulic engineering can be found in [1], which is still one of the most important standard textbooks for the dynamic modeling of hydraulic drive systems. A contemporary extension of this book is represented by [2] with additional prospects regarding modern control engineering. The mentioned literature is very popular in hydraulic engineering, because the presented models give an intuitive access to their dynamic behaviour in an engineering context. In contrast to the fundamentals of fluid mechanics (see, for instance [3]), where the dynamic state of the fluid continuum is represented by density and velocity, their pendants in hydraulic engineering are most often measurable quantities, like pressure and flow rate, which makes the design of hydraulic drive systems quite handsome. For instance, a valve generates a flow rate according to a pressure difference and, a pressure acting on an effective area produces a force, which can be directly used for performing mechanical work in a hydro-mechanical actuator. Due to the fact that pressure and flow rate can be more or less easily measured, both quantities are also called sensor coordinates, which in turn simplifies particularly the model based controller design for hydraulic drive systems. For instance, the potential of non-linear control with respect to input-output behaviour in heavy duty hydraulic drives in rolling mills is clearly demonstrated in [4,5] with physical lumped parameter models. Furthermore, the famous concept of flatness (see, e.g., [6]) has been often successfully applied in hydraulic drive technology as presented in numerous contributions, like [7–9]. In contrast to drive systems with proportional valves or rotary displacement pumps also the flatness based control with digital valves operated in pulse-width-mode (PWM) employs lumped parameter hydraulic models for the controller design, as for instance demonstrated in [10]. Another example is the optimal feedforward control of a hydro-mechanical actuator using a digital switching strategy with directional valves as presented in [11].

In recent years, growing interest in structure-preserving and energy-based modeling frameworks (see, for instance [12–16]) has further motivated a re-examination of lumped parameter formulations for hydraulic systems, particularly with regard to their consistency, passivity properties, and suitability for model-based control design like presented in [17–20] for proportional valve controlled (servo-hydraulic) systems. Moreover, the energy-based concept has been successfully applied in PWM controlled digital hydraulic drives as well, like presented in [21].

Most of the mentioned examples represent so called open hydraulic systems, which means that fluid for driving operations is usually taken from a main (large) tank, then pressurized by a pump and after mechanical work has been performed in the actuator, the oil is returned to the main tank again. In such drive systems at moderate simulation periods the model representation in sensor coordinates works properly. But in closed systems, like hydraulic dampers, or also in long-term simulations the description of the dynamic states by pressure and flow rate is not satisfying anymore, because the conventional lumped models based on volume flow suffer from a small but considerable defect of the mass balance. This effect was already found and discussed in [22,23], leading to a call for further development of simulation tools to achieve a correct mass-conservative approach. Based on these two references in [24] a lumped thermodynamic model of a resistance for different temperatures was investigated, however, for the sake of simplicity the considerations in the present paper are limited to adiabatic reversible processes. Some state of the art simulation tools use certain DAE solvers for a numerical correction of the mass balance and other tools compensate the error in the mass flow by means of a specific correction factor in the numerical simulations. But the basic models are still showing the initial physical inconsistency. However, in certain components the energy balance is even violated. Consequently, a meaningful interpretation of the physical behaviour of the models is no longer possible. For this reason in this contribution the shortcomings of the

established lumped parameter models are investigated systematically and, furthermore, an analytic method for the correction of the inconsistencies in mass- and energy balance is presented. The investigations result in a better understanding of the current models, which in turn facilitates the development of energy-based controllers based on simple models that nevertheless require physical consistency. Moreover, it is shown that the mass- and energy balance can be maintained by simple amendments in existing simulation tools. This enables physically plausible simulations to be achieved in a straightforward manner, particularly for closed hydraulic drive systems. Regarding widely used lumped parameter pipeline models based on RLC elements selected simulations demonstrate that their physical inconsistencies play only a minor role when compared with the corresponding corrected models. It should be noted, however, that the objective of this publication is not to demonstrate that the corrected models provide a better representation of reality. Rather, it can be concluded that the existing RLC models, despite their shortcomings, do not immediately lose their physical relevance even in low pressure simulations.

The paper is structured as follows; the remaining part of the introduction is devoted to common hydraulic (dynamic) models, to basics from thermodynamics and to the framework for energy-based modeling with regard to physical systems modeling necessary for the subsequent considerations below. In Section 2 the limits of the conventional modeling approach suffering from deficiencies in the mass balance are presented by the example of a simple hydraulic damper and, furthermore, how the model can be corrected in order to fulfill the requirements for a mass-conservative behaviour. In Section 3 the energy balance in the dynamic model of an extended hydraulic damper is investigated. Then the findings from the previous sections regarding mass- and energy-balance are used in Section 4 for the investigation of a dynamic pipeline model based on lumped RLC elements. The simulations in this study were carried out in python using the algorithm *solve_ivp* from the *scipy* package [25]. All models presented in this paper were tested with all solvers (RK23, RK45, DOP853, Radau, BDF, and LSODA) available in *solve_ivp*. Only marginal numerical differences were observed, depending on the tolerance settings of the solvers. For this reason, further details regarding numerical accuracy are omitted throughout the manuscript for the sake of simplicity. However, it is important to note that identical solver settings are used whenever simulation results are compared in the paper to ensure a meaningful comparison of the different models. Following a detailed discussion of the major findings in a separate section, the paper concludes with a summary of the main results of the investigation.

1.1. Established Hydraulic Lumped Parameter Models

The set of lumped-parameter models considered in this paper consists primarily of hydraulic resistances, capacitances, and inductances, which represent the hydraulic counterparts of their electrical-domain equivalents. Additionally, in hydraulics models for the fluid and for a hydro-mechanical actuator are necessary. The lumped parameter models used in this contribution can be also found in the established literature (see for instance [1,2,26]) and, are presented in the following as a starting point and for the reader's convenience.

The basic equations of motion for the piston's position x and velocity v of a hydro-mechanical actuator with two pressure chambers driving a dead load m read

$$\dot{x} = v \quad (1)$$

$$\dot{v} = \frac{1}{m}(p_1 A_1 - p_2 A_2 - d_v v + F_L) \quad (2)$$

with the load force F_L , the hydraulic forces $p_i A_i$, $i = 1, 2$ resulting from pressures p_i and the effective hydraulic areas A_i . For simplicity, only a velocity depending viscous friction with coefficient d_v is considered.

The fluid used in oil hydraulics for power transmission is fairly incompressible, however, in dynamic models the compressibility must be taken into account. For this purpose the first-order Taylor series approximation of the equation of state of a liquid is investigated at a constant temperature T_0 , which reads

$$\rho = \rho_0 + \left. \frac{\partial \rho}{\partial p} \right|_{T_0} (p - p_0) \quad (3)$$

with the fluid density ρ at the pressure p and ρ_0 , p_0 as the density and pressure at a reference condition. Equation (3) can be rewritten to the relative change of the density with respect to the pressure p , which follows to

$$\frac{\rho - \rho_0}{\rho} = \frac{1}{\rho} \frac{\partial \rho}{\partial p} (p - p_0) = \frac{1}{\beta} (p - p_0).$$

The relation

$$\frac{1}{\rho} \frac{\partial \rho}{\partial p} \equiv \frac{1}{\beta} \quad (4)$$

represents the definition of a constitutive law for the compressibility of the fluid with the bulk modulus β . The density with respect to pressure p follows after a separation of the variables to

$$\rho = \rho_0 e^{\int_{p_0}^p \frac{1}{\beta} dp} = \rho_0 e^{\frac{p-p_0}{\beta}}. \quad (5)$$

With the fluid model (4) the pressure build up equation in a fluid-filled cavity due to the sum of the incoming and outgoing volumetric flow rates q_i reads

$$\dot{p} = \frac{1}{C_h} \sum q_i \quad (6)$$

with the pressure state p and the hydraulic capacitance $C_h = \frac{V}{\beta}$ as the fraction of the cavity volume V and the bulk modulus of the fluid β .

The common dynamic model for the laminar flow q through a lumped inductance is based on the momentum equation of the fluid inertia in a pipe and calculates to

$$\dot{q} = \frac{1}{L_h} (p_1 - p_2 - R_h q) \quad (7)$$

with the hydraulic inductance $L_h = \frac{\rho_0 l_p}{r_p^4 \pi}$, the laminar flow resistance according to Hagen-Poiseuille $R_h = \frac{8 \rho_0 \nu l_p}{r_p^4 \pi}$ and the input pressures p_1 and p_2 at both ends of the pipe. The dimensions of the pipe are represented by l_p as the corresponding pipe length and r_p as the inner pipe radius. Furthermore, the kinematic viscosity ν represents the major friction coefficient of the fluid.

The flow rate through a hydraulic laminar resistance R_h reads

$$q = \frac{p_1 - p_2}{R_h} \quad (8)$$

and the model of an orifice is given by the well known square-root characteristics

$$q = \alpha A_O \sqrt{\frac{2}{\rho}} \sqrt{p_1 - p_2} \quad (9)$$

with the contraction coefficient α and the cross-sectional area A_O of the orifice.

At this point it should be noted that, although the models (7)–(9) have two ports (p_1, p_2) , only one state, respectively, output q is defined, which implicates that these models are based on an incompressible modeling approach.

1.2. Basics from Thermodynamics

Since the fluid is compressible, energy is stored due to a compression and released during an expansion. Therefore, from thermodynamics (see for instance [27]) the change of the specific internal energy due to the change of the specific volume $v_s = 1/\rho$ of a substance reads

$$du = -pdv_s = -p d\left(\frac{1}{\rho}\right) = \frac{p}{\rho^2} d\rho \quad (10)$$

under the assumption of an adiabatic process. Following, the internal energy stored in the substance with mass M calculates to

$$U = Mu = -M \int_{v_s} p(\vartheta) d\vartheta = M \int_{\rho} \frac{p(\varrho)}{\varrho^2} d\varrho. \quad (11)$$

Another important thermodynamic quantity is the enthalpy

$$H = U + pV \quad (12)$$

which is defined as the sum of the internal energy and the pressure-volume work required for the fluid transport. Furthermore, the process is assumed to be lossless reversible and, thus, it is called isentropic. Consequently, the relations for the specific enthalpy read

$$dh = \underbrace{du + pdv_s}_{=0} + \frac{1}{\rho} dp \quad \text{and} \quad h = u + \frac{1}{\rho} p = u + \frac{1}{\rho^2} \rho^2 \frac{\partial u}{\partial \rho} = \frac{\partial}{\partial \rho}(u\rho). \quad (13)$$

1.3. Energy-Based Modeling

Port-Hamiltonian systems are a modeling framework (see for instance [12,15]) that generalizes classical Hamiltonian mechanics to open systems, which exchange energy with their environment through well-defined ports, like inputs and outputs. In the context of hydro-mechanical drive systems the structure of the dynamics with the state $\mathbf{x} \in \mathbb{R}^{\varkappa}$ with the dimension $\varkappa$ is characterized by three ingredients: the energy storage function $E(\mathbf{x})$ (also known as the Hamiltonian), a skew-symmetric interconnection matrix $\mathbf{J}(\mathbf{x})$ capturing lossless energy exchange between internal components, and a positive semi-definite matrix $\mathbf{R}(\mathbf{x})$ accounting for energy losses, which result in

$$\dot{\mathbf{x}} = (\mathbf{J}(\mathbf{x}) - \mathbf{R}(\mathbf{x})) \left(\frac{\partial E(\mathbf{x})}{\partial \mathbf{x}} \right)^T + \mathbf{g}(\mathbf{x}) \mathbf{u} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x}) \mathbf{u}, \quad \mathbf{y} = \mathbf{g}(\mathbf{x})^T \left(\frac{\partial E(\mathbf{x})}{\partial \mathbf{x}} \right)^T \quad (14)$$

with the input matrix $\mathbf{g}(\mathbf{x})$ and the input vector $\mathbf{u}$. Furthermore, if the change of the energy along trajectories is negative semi-definite, like

$$\frac{d}{dt} E(\mathbf{x}) = \frac{\partial E(\mathbf{x})}{\partial \mathbf{x}} (\mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x}) \mathbf{u}) = \mathbf{y}^T \mathbf{u} - \left(\frac{\partial E(\mathbf{x})}{\partial \mathbf{x}} \right) \mathbf{R}(\mathbf{x}) \left(\frac{\partial E(\mathbf{x})}{\partial \mathbf{x}} \right)^T \leq \mathbf{y}^T \mathbf{u}, \quad (15)$$

then the system is called passive, which is a basic property of physical systems and means that the energy stored in the system must be lower or equal the amount of energy, which is put into the system through its ports.

Moreover, in hydro-mechanical systems there exist so called Casimir functions $\mathbf{C}(\mathbf{x})$ representing dynamical invariants, which are independent of the Hamiltonian E and, thus, fulfill the following set of partial differential equations

$$\frac{\partial \mathbf{C}(\mathbf{x})}{\partial \mathbf{x}} \mathbf{J}(\mathbf{x}) = \mathbf{0} \quad (16)$$

resulting with the *canonical* coordinates $\mathbf{z} = \Phi(\mathbf{x}) = [x, mv, C_1(\mathbf{x}), \dots, C_k(\mathbf{x})]^T$ under $k = \varkappa - 2$ in

$$\dot{\mathbf{z}} = \left(\mathbf{J}^\natural(\mathbf{z}) - \mathbf{R}^\natural(\mathbf{z}) \right) \left(\frac{\partial E^\natural(\mathbf{z})}{\partial \mathbf{z}} \right)^T + \mathbf{g}^\natural(\mathbf{z}) \mathbf{u} \quad (17)$$

with

$$E^\natural(\mathbf{z}) = E(\Phi^{-1}(\mathbf{z})) = E(\mathbf{x}) \quad (18)$$

$$\mathbf{J}^\natural(\mathbf{z}) = \left(\frac{\partial \Phi(\mathbf{x})}{\partial \mathbf{x}} \right) \mathbf{J}(\mathbf{x}) \left(\frac{\partial \Phi(\mathbf{x})}{\partial \mathbf{x}} \right)^T \Big|_{\mathbf{x}=\Phi^{-1}(\mathbf{z})} \quad (19)$$

$$\mathbf{R}^\natural(\mathbf{z}) = \left(\frac{\partial \Phi(\mathbf{x})}{\partial \mathbf{x}} \right) \mathbf{R}(\mathbf{x}) \left(\frac{\partial \Phi(\mathbf{x})}{\partial \mathbf{x}} \right)^T \Big|_{\mathbf{x}=\Phi^{-1}(\mathbf{z})} \quad (20)$$

$$\mathbf{g}^\natural(\mathbf{z}) = \left(\frac{\partial \Phi(\mathbf{x})}{\partial \mathbf{x}} \right) \mathbf{g}(\mathbf{x}) \Big|_{\mathbf{x}=\Phi^{-1}(\mathbf{z})} \quad (21)$$

and the Jacobian $\frac{\partial \Phi(\mathbf{x})}{\partial \mathbf{x}}$ (see, for instance [28]).

This structure is convenient for modeling physical systems because it directly encodes energy balance: the change in stored energy equals the power flowing in through the ports reduced by the power dissipated internally. Since energy conservation and passivity are fundamental properties of real physical systems (in this specific case mechanical systems and thermodynamic processes), a port-Hamiltonian formulation automatically respects these physical laws by construction, rather than imposing them as external constraints. This also makes it very natural to interconnect subsystems from different physical domains, in particular hydro-mechanical systems, while preserving overall energy consistency.

2. A Simple Hydraulic Damper

In Figure 1a a closed hydraulic system comprising a double rod cylinder ($A_1 = A_2 = A$) with the actuator length l and a laminar resistance R_h located between both pressure chambers representing a hydraulic damper is illustrated.

With the hydraulic and mechanical models presented in the introduction the dynamic system according to the conventional modeling approach results in

$$\begin{bmatrix} \dot{x} \\ \dot{v} \\ \dot{p}_1 \\ \dot{p}_2 \end{bmatrix} = \begin{bmatrix} v \\ \frac{1}{m}((p_1 - p_2)A - d_v v + F) \\ \frac{\beta}{Ax} \left(-Av - \frac{p_1 - p_2}{R_h} \right) \\ \frac{\beta}{A(l-x)} \left(Av + \frac{p_1 - p_2}{R_h} \right) \end{bmatrix} = \mathbf{f}_{\mathcal{R}}(\mathbf{x}) + \mathbf{g}(\mathbf{x})F \quad (22)$$

with the state $\mathbf{x} = [x, v, p_1, p_2]^T$ containing x and v for position and velocity and the pressures p_1 and p_2 in both chambers of the actuator. The input is given by the force F . Since the entries of the state $\mathbf{x}$ represent basically measurable quantities, they are also called sensor coordinates. The simulation results for a parameter set with regard to Table 1 and due to a rectangular input force F are depicted in Figure 1b. It is remarkable that at the end of the simulation period the pressure in both chambers is obviously lower than the initial pressure, even though no external leakage is modelled. Possible reasons for

this behaviour are: either the mass balance is violated and/or the energy in the system is unbalanced. Therefore, in the following the mass and energy conservation in system (22) are investigated.

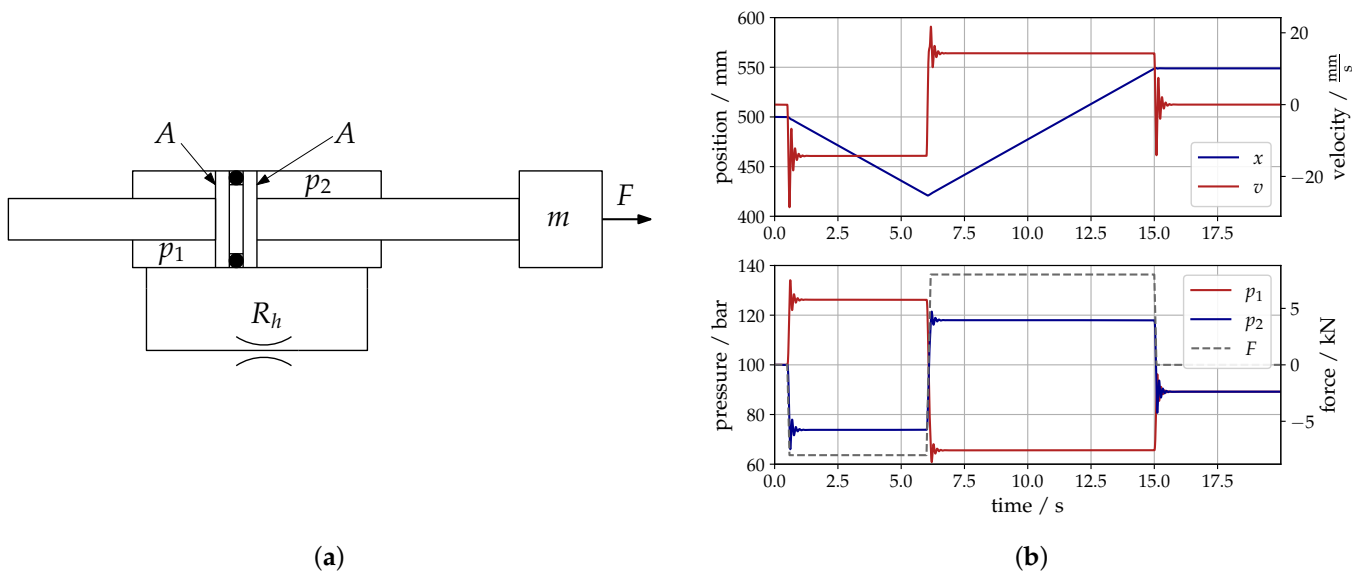


Figure 1. A simple hydraulic damper. (a) Functional scheme of the damper; (b) Simulation results.

Table 1. Parameters for simulation.

Parameter	Value
dead mass	$m = 5000 \text{ kg}$
cross-sectional area	$A = (63^2 - 45^2) \pi / 4 \text{ mm}^2$
maximum actuator stroke	$l = 1 \text{ m}$
viscous friction	$d_v = 0 \frac{\text{Ns}}{\text{m}}$
laminar resistance	$R_h = 40 \frac{\text{bar}}{\ell/\text{min}}$
reference pressure	$p_0 = 1 \text{ bar}$
density at reference pressure	$\rho_0 = 860 \frac{\text{kg}}{\text{m}^3}$
bulk modulus	$\beta = 1.6 \times 10^9 \text{ Pa}$
kinematic viscosity	$\nu = 45 \text{ cSt}$

2.1. Energy-Based Analysis

The Hamiltonian E , respectively, the energy stored in the system from Figure 1a is the sum of the kinetic energy T of the dead load m and the internal energy of the fluid in both pressure chambers $V_1 = Ax$ and $V_2 = A(l - x)$, which calculates by using Equation (11) to

$$\begin{aligned}
 E &= T + U_1 + U_2 = \\
 &= \frac{mv^2}{2} + V_1 \left((\beta + p_0) e^{\frac{p_1 - p_0}{\beta}} - \beta - p_1 \right) + V_2 \left((\beta + p_0) e^{\frac{p_2 - p_0}{\beta}} - \beta - p_2 \right). \quad (23)
 \end{aligned}$$

In a first step the flow between the chambers is impeded ($R_h \rightarrow \infty$) resulting in the simplified system dynamics

$$\begin{bmatrix} \dot{x} \\ \dot{v} \\ \dot{p}_1 \\ \dot{p}_2 \end{bmatrix} = \begin{bmatrix} v \\ \frac{1}{m}((p_1 - p_2)A - d_v v + F) \\ -\frac{\beta}{Ax}Av \\ \frac{\beta}{A(l-x)}Av \end{bmatrix} = \mathbf{f}_0(\mathbf{x}) + \mathbf{g}(\mathbf{x})F, \quad (24)$$

which can be written in PCHD structure from Equation (14) with

$$\left(\frac{\partial E(\mathbf{x})}{\partial \mathbf{x}}\right)^T = \begin{bmatrix} A\left(p_2 - p_1 + (\beta + p_0)\left(e^{\frac{p_1 - p_0}{\beta}} - e^{\frac{p_2 - p_0}{\beta}}\right)\right) \\ mv \\ -\frac{\left(\beta - (\beta + p_0)e^{\frac{p_1 - p_0}{\beta}}\right)V_1}{\beta} \\ -\frac{\left(\beta - (\beta + p_0)e^{\frac{p_2 - p_0}{\beta}}\right)V_2}{\beta} \end{bmatrix} \quad (25)$$

and the matrices

$$\mathbf{J}_0(\mathbf{x}) = \begin{bmatrix} 0 & \frac{1}{m} & 0 & 0 \\ -\frac{1}{m} & 0 & \frac{\beta}{mx} & -\frac{\beta}{m(l-x)} \\ 0 & -\frac{\beta}{mx} & 0 & 0 \\ 0 & \frac{A\beta}{m(l-x)} & 0 & 0 \end{bmatrix} \text{ and } \mathbf{R}_0(\mathbf{x}) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{d_v}{m^2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (26)$$

following in the negative semi-definite passivity condition

$$\frac{d}{dt}E \underset{\substack{\uparrow \\ F=0}}{=} \frac{\partial E}{\partial \mathbf{x}} \mathbf{f}_0(\mathbf{x}) = -\left(\frac{\partial E(\mathbf{x})}{\partial \mathbf{x}}\right) \mathbf{R}_0(\mathbf{x}) \left(\frac{\partial E(\mathbf{x})}{\partial \mathbf{x}}\right)^T = -d_v v^2 \leq 0, \quad (27)$$

which in turn means that in case of $d_v > 0$ energy is dissipated within the system and, thus, the system shows an energy-consistent behaviour. Furthermore, by solving Equation (16) the dynamic invariants of the system can be identified as the fluid masses in the pressure chambers

$$C_i(\mathbf{x}) = M_i(x, p_i) = \rho_i(p_i)V_i(x), \quad i = 1, 2, \quad (28)$$

representing with x and mv the canonical coordinates $\mathbf{z}$.

2.2. Mass Balance

The fundamental mass balance equation

$$\dot{M} = \rho q_{in} \quad (29)$$

with the fluid mass $M = \rho V$ describes the change of mass in a volume V due to volumetric flow q_{in} . This is taken as a starting point for the derivation of the well known and established pressure build up equation. The relations follow to

$$\begin{aligned} \dot{M} &= \frac{d(\rho V)}{dt} = \rho q_{in} \\ \dot{\rho}V + \rho\dot{V} &= \rho q_{in} \\ \frac{\partial \rho}{\partial p} \frac{dp}{dt} V + \rho\dot{V} &= \rho q_{in} \end{aligned}$$

and finally dividing by ρ yields

$$\underbrace{\frac{1}{\rho} \frac{\partial \rho}{\partial p}}_{\kappa(p)} \dot{p} V + \dot{V} = q_{in} \quad (30)$$

the pressure build up equation for p in a volume V according to the constitutive material law of the fluid

$$\frac{1}{\rho} \frac{\partial \rho}{\partial p} = \kappa(p) \quad (31)$$

and the volumetric input flow rate q_{in} . Using now the relation for the density from Equation (5) and the flow rate through the laminar resistance from Equation (8) in system (22) yields the mass flows into both pressure chambers of the hydraulic damper from Figure 1a, which read

$$\dot{M}_1 = -\rho_1 q = -\rho_0 e^{\frac{p_1 - p_0}{\beta}} \frac{(p_1 - p_2)}{R_h} \quad (32)$$

$$\dot{M}_2 = \rho_2 q = \rho_0 e^{\frac{p_2 - p_0}{\beta}} \frac{(p_1 - p_2)}{R_h}. \quad (33)$$

Due to $\dot{M}_1 \neq -\dot{M}_2$ the mass balance between both chambers is obviously violated, which seems to be the reason for the pressure loss in the simulation results from Figure 1b. In order to compensate the unbalance in the mass flow, the input flow rate q_{in} in the pressure build up equation must be written as a mass flow $\dot{M}_{in}$ divided by the density ρ , like

$$\kappa(p) \dot{p} V + \dot{V} = q_{in} = \frac{\dot{M}_{in}}{\rho} \underset{\substack{\uparrow \\ \text{Equation (5)}}}{=} \underbrace{\frac{\dot{M}_{in}}{\rho_0}}_{q_{in}|_{p_0}} \underbrace{e^{-\int_{p_0}^p \kappa(\tilde{p}) d\tilde{p}}}_{\chi^{-1}} \quad (34)$$

with

$$\chi = \frac{\rho}{\rho_0} = e^{\int_{p_0}^p \kappa(\tilde{p}) d\tilde{p}} \quad (35)$$

as the *compression ratio* of the density due to a compression $p_0 \rightarrow p$. The right hand side of Equation (34) can be interpreted as an incompressible flow $q_{in}|_{p_0} = \frac{\dot{M}_{in}}{\rho_0}$ at reference pressure p_0 multiplied by the inversed compression ratio χ^{-1} with regard to the relevant pressure p . Thus, the corrected flow rate through a laminar resistance using the compression ratio χ from Equation (35) and the fluid model (4) results in the corrected pressure build up equations

$$\begin{aligned} \frac{V_1}{E} \dot{p}_1 + A v &= q_1 = -\frac{(p_1 - p_2)}{R_h} \underbrace{e^{-\frac{p_1 - p_0}{\beta}}}_{\chi_1^{-1}} \\ \frac{V_2}{E} \dot{p}_2 - A v &= q_2 = \frac{(p_1 - p_2)}{R_h} \underbrace{e^{-\frac{p_2 - p_0}{\beta}}}_{\chi_2^{-1}}, \end{aligned} \quad (36)$$

which now fulfill the mass balance due to

$$\dot{M}_1 = -\frac{\rho_1}{\chi_1} q = -\rho_0 \frac{(p_1 - p_2)}{R_h} \quad (37)$$

$$\dot{M}_2 = \frac{\rho_2}{\chi_2} q = \rho_0 \frac{(p_1 - p_2)}{R_h}. \quad (38)$$

The derivative with respect to time of the storage function follows to

$$\frac{d}{dt} E \underset{\substack{\uparrow \\ F=0}}{=} \frac{\partial E}{\partial \mathbf{x}} \mathbf{f}_{\mathcal{R}}(\mathbf{x}) = -d_v v^2 - \frac{\beta}{R_h} (p_1 - p_2) \left(e^{-\frac{p_2}{\beta}} - e^{-\frac{p_1}{\beta}} \right) e^{\frac{p_0}{\beta}} \leq 0, \quad (39)$$

which is negative semi-definite since both brackets have always the same sign and, thus, an energy-consistent behaviour of the system can be confirmed with

$$\mathbf{J}_{\mathcal{R}}(\mathbf{x}) = \begin{bmatrix} 0 & \frac{1}{m} & 0 & 0 \\ -\frac{1}{m} & 0 & \frac{\beta}{mx} & -\frac{A\beta}{m(l-x)} \\ 0 & -\frac{\beta}{mx} & 0 & 0 \\ 0 & \frac{\beta}{m(l-x)} & 0 & 0 \end{bmatrix}, \mathbf{R}_{\mathcal{R}}(\mathbf{x}) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{d_v}{m^2} & 0 & 0 \\ 0 & 0 & \frac{\zeta}{V_1^2} \frac{\chi_2}{\chi_1} & -\frac{\zeta}{V_1 V_2} \\ 0 & 0 & -\frac{\zeta}{V_1 V_2} & \frac{\zeta}{V_2^2} \frac{\chi_1}{\chi_2} \end{bmatrix} \quad (40)$$

and $\zeta = \frac{\beta}{R_h} \frac{(p_1 - p_2)}{(\chi_1 - \chi_2)} \geq 0$. Using the state transformation

$$\mathbf{z} = \Phi(\mathbf{x}) = \begin{bmatrix} x \\ mv \\ \rho_0 e^{\frac{p_1 - p_0}{\beta}} V_1 \\ \rho_0 e^{\frac{p_2 - p_0}{\beta}} V_2 \end{bmatrix} = \begin{bmatrix} x \\ mv \\ M_1 \\ M_2 \end{bmatrix} \quad (41)$$

and its Jacobian

$$\frac{\partial \Phi(\mathbf{x})}{\partial \mathbf{x}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & m & 0 & 0 \\ A\rho_0 e^{\frac{p_1 - p_0}{\beta}} & 0 & \frac{\rho_0 V_1 e^{\frac{p_1 - p_0}{\beta}}}{\beta} & 0 \\ -A\rho_0 e^{\frac{p_2 - p_0}{\beta}} & 0 & 0 & \frac{\rho_0 V_2 e^{\frac{p_2 - p_0}{\beta}}}{\beta} \end{bmatrix} \quad (42)$$

results in a representation in canonical coordinates

$$\mathbf{J}_{\mathcal{R}}^{\natural}(\mathbf{z}) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \mathbf{R}_{\mathcal{R}}^{\natural}(\mathbf{z}) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & d & 0 & 0 \\ 0 & 0 & \delta & -\delta \\ 0 & 0 & -\delta & \delta \end{bmatrix} \quad (43)$$

with

$$\delta = \frac{\rho_0 M_1 M_2 \ln\left(\frac{M_1 V_2}{M_2 V_1}\right)}{R_h (M_2 V_1 - M_1 V_2)} \underset{\mathbf{z} = \Phi(\mathbf{x})}{=} \zeta \frac{\rho_1 \rho_2}{\beta^2} = \frac{\rho_1 \rho_2}{R_h \beta} \frac{(p_1 - p_2)}{(\chi_1 - \chi_2)} = \frac{\rho_0}{R_h} \frac{p_1 - p_2}{h_1 - h_2} \geq 0 \quad (44)$$

and

$$h_i = \frac{\partial}{\partial \rho_i} (u_i \rho_i) = \frac{\partial}{\partial \rho_i} \left(\frac{U_i}{V_i} \right) = \frac{\beta + p_0}{\rho_0} - \frac{\beta}{\rho_i} \quad (45)$$

representing the enthalpy from Equation (13) in each pressure chamber.

In Figure 2 the simulation results with the mass balanced pressure build up equations from Equation (36) are depicted. Now the final pressure values coincide with the initial ones as expected. It is noteworthy that compared to the results from Figure 1b the pressure signals in Figure 2 show an obviously constant slope during the movement at constant velocity, which seems to be an effect of the balanced mass flow resulting in $A_i v - \frac{q_i}{\chi_i} \neq 0 \rightarrow \dot{p}_i \neq 0$ (see also Equation (36)) in combination with the changing volumes V_i of the pressure chambers.

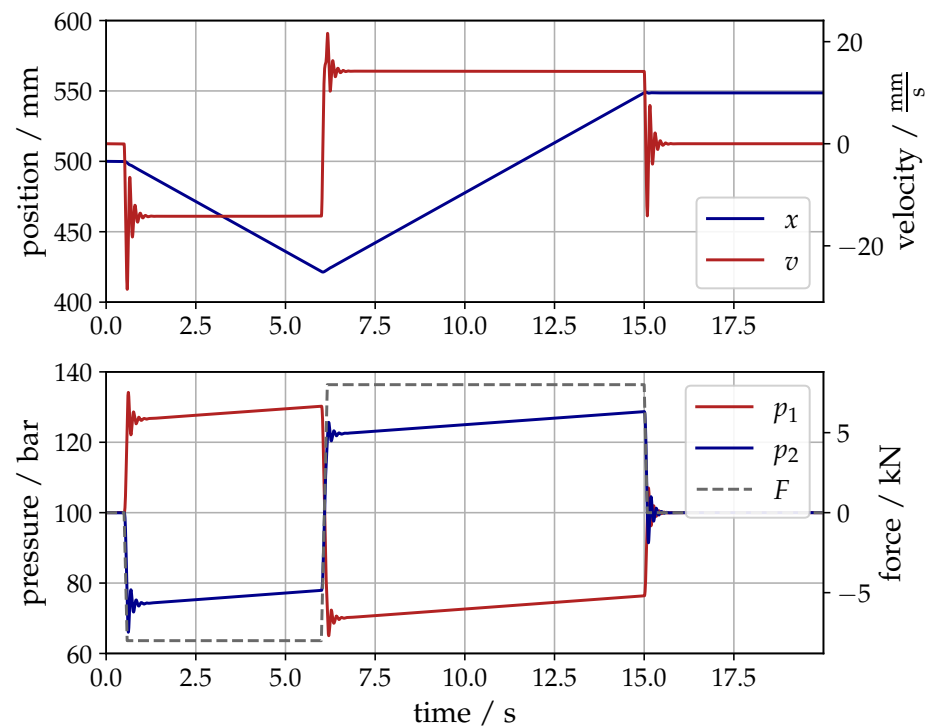


Figure 2. Simulation results for the damper with mass balance correction.

2.3. Mass Flow Compensation

The unbalanced mass flow in common static resistance models from Equation (8) is quite small and does not play a significant role in open hydraulic systems directing the fluid from a constant supply pressure over a certain load to an ideal tank. Since the effect must be considered only in a rare number of dynamic simulations of closed hydraulic systems, a separate simulation toolbox is not necessary. The mass flow compensation can be easily realized by additional basic simulation blocks, like illustrated in Figure 3.

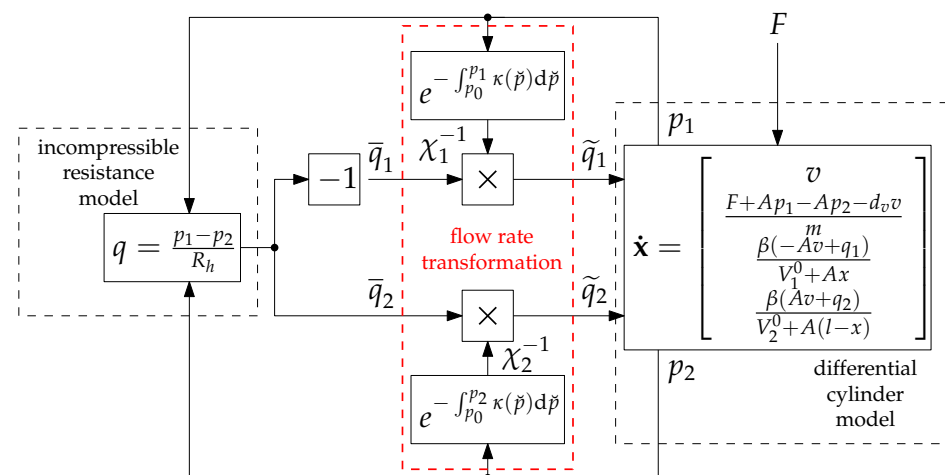


Figure 3. Mass conservation in existing simulation tools.

Furthermore, this compensation is also applicable with other fluid models like, for instance, an elastic liquid with entrained air (see for instance [29–33]) with the constitutive law

$$\kappa(p) = \frac{1 - \varphi_0}{\beta_0} + \frac{\varphi_0}{np} = \frac{1}{\beta_{\text{eff}}(p)} \quad (46)$$

with a volume fractional amount of gas $\varphi_0 = \frac{V_{\text{gas}}}{V_{\text{liquid}}}$ according to a polytropic change of state with exponent n in the elastic liquid with a constant bulk modulus β_0 . The effective bulk modulus β_{eff} is now pressure dependent as shown in Figure 4a and the compression ratio simply calculates to

$$\chi = e^{\int_{p_0}^p \kappa(\check{p}) d\check{p}} = \left(\frac{p}{p_0} \right)^{\frac{\varphi_0}{n}} e^{\frac{p-p_0}{\beta_0} (1-\varphi_0)}. \quad (47)$$

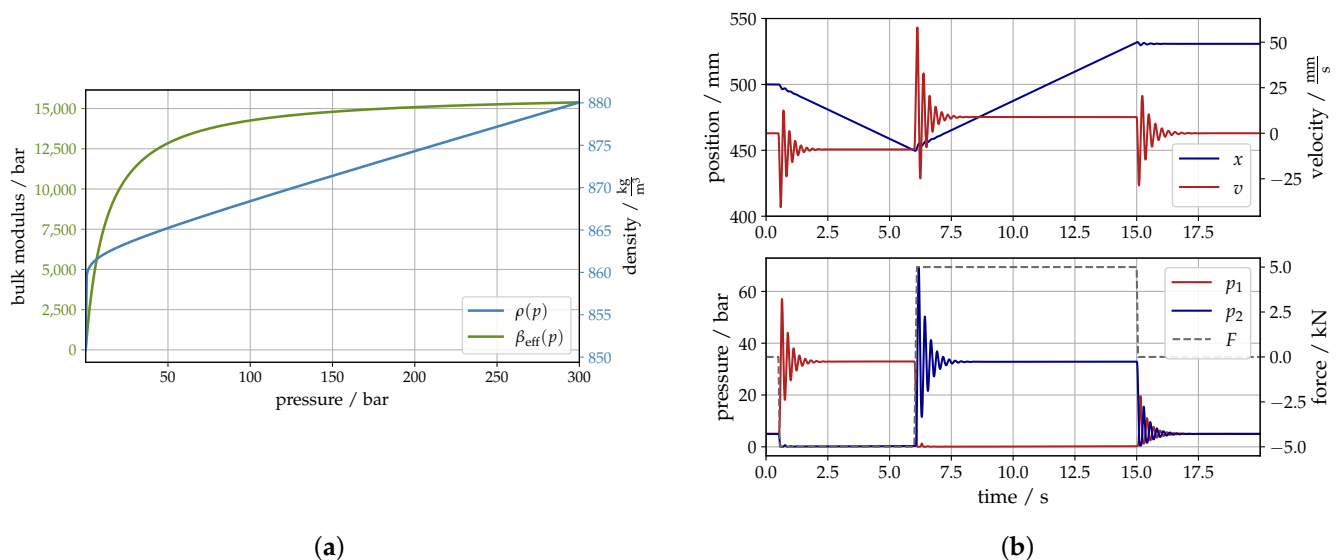


Figure 4. Simulations with an advanced fluid model with entrained air. (a) Fluid model (46) with $\varphi_0 = 0.001$ and $n = 1.3$; (b) Simulation results.

The corresponding simulation results from Figure 4b show that the mass flow compensation works properly even at very low pressure levels. It must be remarked that for pressures tending to zero also the fluid density tends to zero, however, the compressibility law from Equation (46) does not reflect the effect of cavitation at all, rather the expansion of the uniformly distributed gas bubbles leads to this behaviour, which in turn represents a lower limit for the pressure values. However, if an analytic solution of the integral $\int_{p_0}^p \kappa(\check{p}) d\check{p}$ in the compression ratio χ exists, then the mass flow compensation can be realized as a simple static function block and is easily applicable to existing simulation tools. For other, eventually, more advanced fluid models a numerical approximation of the integral in the compression ratio might become necessary, which may lead to considerable extended calculation times, since in time domain simulations the compression ratio must be evaluated multiple times by the solver at each time step.

For the fluid model (46) the energy storage function E must be adapted due to the pressure dependence of $\kappa(p) = \frac{1}{\beta_{\text{eff}}(p)}$ for both individual chambers, which results in the negative semi-definite relation

$$\frac{d}{dt} E \Big|_{F=0} = \frac{\partial E}{\partial \mathbf{x}} \mathbf{f}_{\mathcal{R}}(\mathbf{x}) = -d_v v^2 - \frac{\rho_0}{R_h} (p_1 - p_2)(h_1 - h_2) \leq 0 \quad (48)$$

with the specific enthalpy $h_i = \int_{p_0}^{p_i} \frac{d\check{p}}{\rho(\check{p})}$, which is monotonically increasing in p_i due to $\frac{dh_i}{dp_i} = \frac{1}{\rho_i} > 0$ for $i = 1, 2$. With $\varphi_0 = 0$ the passivity condition (48) simplifies to the same result presented in Equation (39), which is a nice cross-check for the correctness of the calculations.

Furthermore, the mass flow compensation can be easily applied to other static flow models as well, like orifices or pump models as, for instance, depicted in Figure 5. The corresponding dynamic model for this drive system reads

$$\begin{bmatrix} \dot{x} \\ \dot{v} \\ \dot{p}_1 \\ \dot{p}_2 \end{bmatrix} = \begin{bmatrix} v \\ \frac{1}{m}((p_1 - p_2)A - d_v v) \\ \frac{\beta}{Ax} \left(-Av + \frac{V_D}{2\pi} \frac{\omega}{\chi_1} \right) \\ -\frac{\beta}{A(1-x)} \left(Av - \frac{V_D}{2\pi} \frac{\omega}{\chi_2} \right) \end{bmatrix} \quad (49)$$

with the displacement volume of the pump $V_D = 10 \text{ cm}^3/\text{rev}$ and the angular velocity ω as the input to the system. For simplicity in the simulations depicted in Figure 6 a rectangular input signal ω was chosen, which represents a considerable excitation of the undamped resonating system. For this reason a viscous damping $d_v = 50,000 \frac{\text{Ns}}{\text{m}}$ was used in order to reach a steady state movement within a reasonable period of time. In the specific case without a correct mass compensation ($\chi_1 = \chi_2 = 1$) the simulation results in Figure 6a show even a higher pressure at the end of the simulation period compared to the initial pressure, which would mean that mass is produced in the system. But, if the mass flow is compensated correctly by the compression ratio χ_i , then the simulations deliver meaningful results like depicted in Figure 6b. Again, the obviously constant slope of the pressure signals during the phases of constant velocity is noteworthy, which is a result from the compression and expansion of the fluid at the individual ports of the pump model and the constantly changing volumes in the pressure build up equations, respectively.

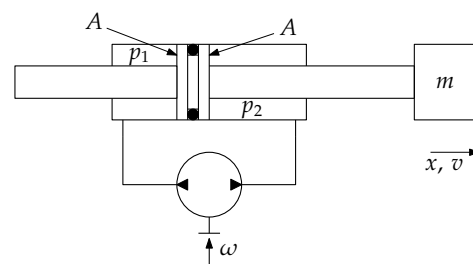


Figure 5. Cylinder with pump.

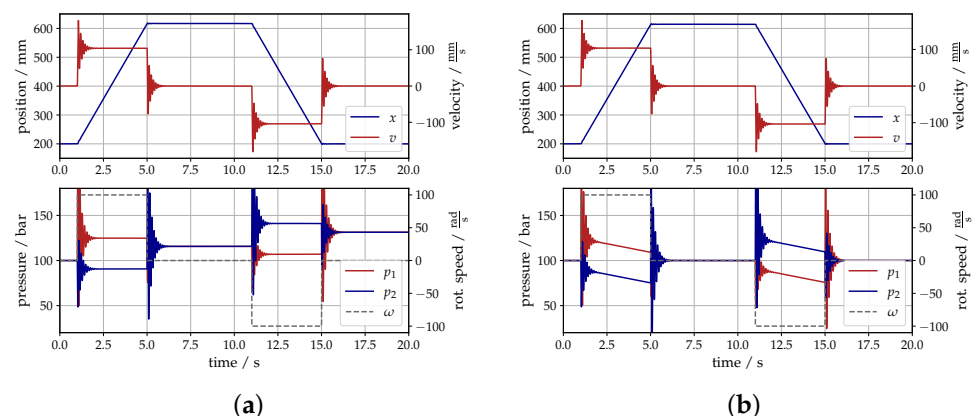


Figure 6. Simulation results for the pump driven cylinder actuator from Equation (49). (a) Without mass flow compensation; (b) With mass flow compensation.

3. Hydraulic Damper with Lumped Inductance

The dynamic model of the system depicted in Figure 7a, where the compression ratios χ_1 and χ_2 are already taken into account, reads

$$\begin{bmatrix} \dot{x} \\ \dot{v} \\ \dot{p}_1 \\ \dot{p}_2 \\ \dot{q} \end{bmatrix} = \begin{bmatrix} v \\ \frac{1}{m}((p_1 - p_2)A - d_v v_P + F) \\ \frac{\beta}{Ax}(-Av - \frac{q}{\chi_1}) \\ -\frac{\beta}{A(l-x)}(Av + \frac{q}{\chi_2}) \\ \frac{1}{L_h}(p_1 - p_2 - R_h q) \end{bmatrix} = \mathbf{f}_{\mathcal{L}}(\mathbf{x}) + \mathbf{g}(\mathbf{x})F \quad (50)$$

with the extended state $\mathbf{x} = [x, v, p_1, p_2, q]^T$, the simple fluid model (4) and the new Hamiltonian

$$E_{\mathcal{L}} = E + \frac{1}{2}L_h q^2. \quad (51)$$

The simulation results presented in Figure 7b for a pipeline with length $l_p = 5$ m and inner pipe radius $r_p = 3$ mm seem to be meaningful since no pressure is lost at the end of the simulation period. But a closer look at the derivative of the Hamiltonian w.r.t. time

$$\frac{d}{dt}E_{\mathcal{L}} \underset{F=0}{=} \frac{\partial E_{\mathcal{L}}}{\partial \mathbf{x}} \mathbf{f}_{\mathcal{L}}(\mathbf{x}) = -d_v v^2 - R_h q^2 + q \left(\beta (\chi_1^{-1} - \chi_2^{-1}) + p_1 - p_2 \right) \underset{F=0}{\geq} 0 \quad (52)$$

shows that due to its indefiniteness according to the last term the lumped inductance model does not represent the expected energy-consistent behaviour. Since the mass balance is already fulfilled by the application of the compression ratios $\chi_i \neq 1$, obviously the energy balance must be violated, most likely due to the fact that the lumped inductance model is derived from an incompressible modeling approach.

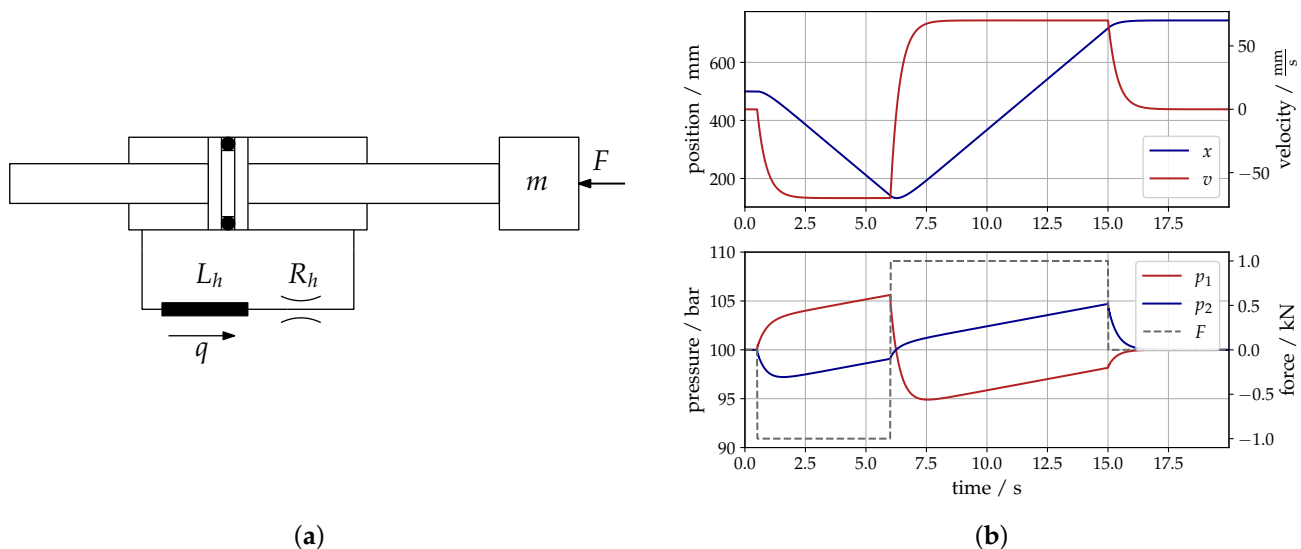


Figure 7. Hydraulic damper with lumped inductance. (a) Functional scheme; (b) Simulation results.

3.1. PCHD-Formulation

In this specific case the Port-Hamiltonian formulation results in the skew-symmetric matrix

$$\mathbf{J}_{\mathcal{L}}(\mathbf{x}) = \begin{bmatrix} 0 & \frac{1}{m} & 0 & 0 & 0 \\ -\frac{1}{m} & 0 & \frac{\beta}{mx} & -\frac{\beta}{m(l-x)} & 0 \\ 0 & -\frac{\beta}{mx} & 0 & 0 & -\frac{\beta}{\chi_1 A L_h x} \\ 0 & \frac{\beta}{m(l-x)} & 0 & 0 & \frac{\beta}{\chi_2 A L_h (l-x)} \\ 0 & 0 & \frac{\beta}{\chi_1 A L_h x} & -\frac{\beta}{\chi_2 A L_h (l-x)} & 0 \end{bmatrix} \quad (53)$$

and the indefinite damping matrix

$$\mathbf{R}_{\mathcal{L}}(\mathbf{x}) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{d_v}{m^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{R_h}{L_h^2} + \frac{(-\beta(\chi_1^{-1} - \chi_2^{-1}) - p_1 + p_2)}{L_h^2 q} \end{bmatrix}. \quad (54)$$

Taking a closer look at the lower right entry in $\mathbf{R}_{\mathcal{L}}(\mathbf{x})$ suggests that the numerator of the second term must be annihilated in order to guarantee positive semi-definiteness. Evaluating the last row of

$$\mathbf{f}_{\mathcal{L}}(\mathbf{x}) = (\mathbf{J}_{\mathcal{L}}(\mathbf{x}) - \mathbf{R}_{\mathcal{L}}(\mathbf{x})) \left(\frac{\partial E_{\mathcal{L}}}{\partial \mathbf{x}} \right)^T \quad (55)$$

results in

$$\frac{1}{L_h}(p_1 - p_2 - R_h q) = \frac{\beta}{\chi_1 A L_h x} \frac{\partial E_{\mathcal{L}}}{\partial p_1} - \frac{\beta}{\chi_2 A L_h (l - x)} \frac{\partial E_{\mathcal{L}}}{\partial p_2} - \mathbf{R}_{\mathcal{L}}[5, 5] \frac{\partial E_{\mathcal{L}}}{\partial q}. \quad (56)$$

Claiming $\mathbf{R}_{\mathcal{L}}[5, 5] = \frac{R_h}{L_h^2}$ for positive semi-definiteness of $\mathbf{R}_{\mathcal{L}}$ requires a modification of the differential equation for the flow rate (left hand side of Equation (56)) like

$$\dot{q} = \frac{1}{L_h} (\widetilde{\Delta p} - R_h q), \quad (57)$$

where the original input $\Delta p = p_1 - p_2$ is replaced by

$$\begin{aligned} \widetilde{\Delta p} &= \frac{\beta}{A x \chi_1} \frac{\partial U_1}{\partial p_1} - \frac{\beta}{A (l - x) \chi_2} \frac{\partial U_2}{\partial p_2} \\ &= \frac{\beta}{\chi_1} \frac{\partial u_1^V}{\partial p_1} - \frac{\beta}{\chi_2} \frac{\partial u_2^V}{\partial p_2} \quad \text{with } u_i^V = u_i / v_i \end{aligned} \quad (58)$$

$$= \left((\beta + p_0) - \frac{\beta}{\chi_1} \right) - \left((\beta + p_0) - \frac{\beta}{\chi_2} \right), \quad (59)$$

which in turn yields

$$\mathbf{R}_{\mathcal{L}}(\mathbf{x}) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{d_v}{m^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{R_h}{L_h^2} \end{bmatrix} \quad (60)$$

and leads to

$$\frac{d}{dt} E_{\mathcal{L}} \underset{\substack{\uparrow \\ F=0}}{=} \frac{\partial E_{\mathcal{L}}}{\partial \mathbf{x}} \mathbf{f}_{\mathcal{L}}(\mathbf{x}) = -d_v v^2 - R_h q^2 \leq 0. \quad (61)$$

In Figure 8 the difference between the original input $p_1 - p_2$ and the modified input from Equation (59) is illustrated for a common pressure range in hydraulic drive systems, where the error for large pressure differences is lower than 1%, however, the error increases with larger pressure differences. Nonetheless, with the presented modifications the mass balance and the energy balance are now fulfilled for a meaningful physical behaviour of the lumped inductance model.

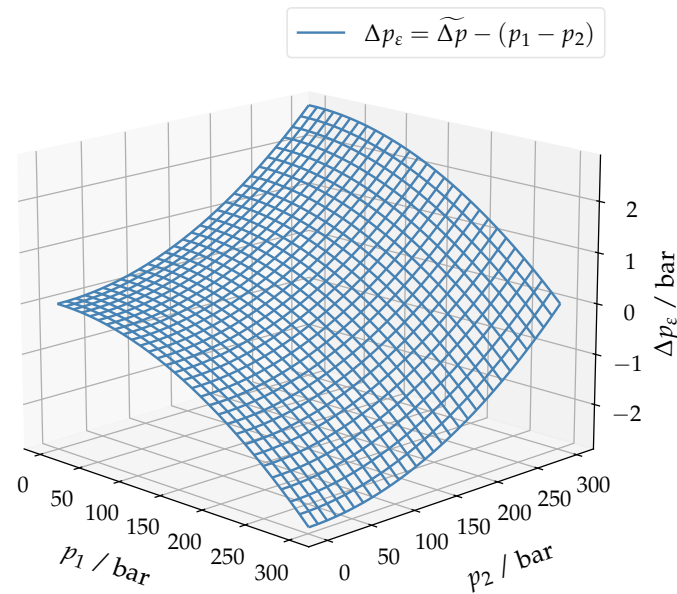


Figure 8. Difference between $p_1 - p_2$ and $\widetilde{\Delta p}$.

3.2. Maintenance of the Energy Balance

The strategy for maintaining the energy balance in lumped inductance models strongly depends on the specific internal energy stored in the fluid, as can be seen from Equation (58), which can be further simplified to

$$\widetilde{\Delta p} = \frac{\beta}{\chi_1} \frac{\partial u_1^V}{\partial p_1} - \frac{\beta}{\chi_2} \frac{\partial u_2^V}{\partial p_2} = \rho_0(h_1 - h_2) = \int_{p_0}^{p_1} \frac{d\check{p}}{\chi_1(\check{p})} - \int_{p_0}^{p_2} \frac{d\check{p}}{\chi_2(\check{p})} = \tilde{p}_1 - \tilde{p}_2. \quad (62)$$

This relation allows a nice interpretation that additionally to the (mechanical) work necessary for transporting the fluid mass through the pipe the driving pressure difference $\widetilde{\Delta p}$ also considers the internal energy according to the compression, respectively, expansion of the fluid entering or leaving the inductance ($h = u + \frac{p}{\rho}$) in order to preserve energy during a flow between both pressure chambers. Furthermore, this is comparable with the result from Equation (48) for the advanced fluid model and, thus, for a pressure dependent compressibility $\kappa(p) = \frac{1}{\beta_{\text{eff}}(p)}$.

Using relation (62), the energy balance for the inductance model can be achieved in existing simulation tools in a similar way compared to the mass flow compensation presented above. Like before with the compression ratio χ , also in this case the necessary correction of the input pressures of the inductance model $\widetilde{\Delta p}$ is feasible by static function blocks, if an analytic solution of $\tilde{p} = \int_{p_0}^p \frac{d\check{p}}{\chi(\check{p})} = \rho_0 h$ exists for the used fluid model, as depicted in Figure 9.

For the advanced fluid model with entrained air from Equation (46) the specific enthalpy calculates to

$$h(p) = \frac{p_0}{\rho_0} + \frac{p_0^{1-s} e^{cp_0} \Gamma(s)}{\rho_0 c^s} (P(s, cp) - P(s, cp_0)) \quad (63)$$

with $s = 1 - \varphi_0/n$, $c = (1 - \varphi_0)/\beta_0$ and $P(s, x) = \gamma(s, x)/\Gamma(s)$ as the regularized lower incomplete Gamma-function, which is available in most simulation environments for an efficient numerical implementation. If no air is entrained in the fluid ($\varphi_0 = 0$), then Equation (63) simplifies again to the relations presented in Equation (59).

From the results presented above it became clear that the mass flow compensation is essential in closed hydraulic systems, but as can be seen from Figure 7b, a violation

of the energy balance is not immediately apparent. This raises the question of whether and how these two effects may also play a role in other lumped hydraulic models. As a class of hydraulic systems with a high number of interconnected lumped parameter models the dynamic model of a hydraulic transmission line based on RLC-elements will be investigated in the following section.

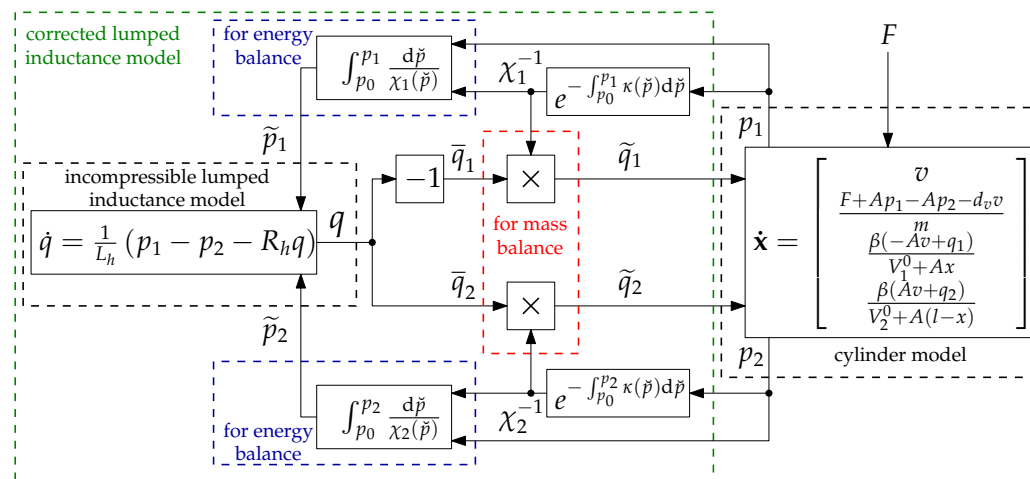


Figure 9. Maintenance of the energy balance for a lumped inductance model in existing simulation tools.

4. Pipeline Model Based on RLC Elements

A common method for an approximation of wave propagation effects in considerably long pipelines is the discretization with lumped RLC-elements, like depicted in Figure 10. Under an application of the results from above the dynamic system for the pipeline follows to

$$\begin{bmatrix} \dot{q}_1 \\ \dot{p}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{p}_{N-1} \\ \dot{q}_N \end{bmatrix} = \begin{bmatrix} \frac{1}{L_1} (\widetilde{\Delta p}_1 - R_1 q_1) \\ \frac{\beta_1}{V_1} (q_1 - q_2) \frac{1}{\chi_1} \\ \frac{1}{L_2} (\widetilde{\Delta p}_2 - R_2 q_2) \\ \vdots \\ \frac{\beta_{N-1}}{V_{N-1}} (q_{N-1} - q_N) \frac{1}{\chi_{N-1}} \\ \frac{1}{L_N} (\widetilde{\Delta p}_N - R_N q_N) \end{bmatrix} = \underbrace{\mathbf{f}_{\text{RLC}}(\mathbf{x})}_{\mathbf{g}_{\text{RLC}}(\mathbf{x})} + \underbrace{\begin{bmatrix} \frac{1}{L_1} & 0 \\ 0 & \vdots \\ \vdots & 0 \\ 0 & \frac{1}{L_N} \end{bmatrix}}_{\mathbf{g}_{\text{RLC}}(\mathbf{x})} \underbrace{\begin{bmatrix} \rho_0 h_0(p_0) \\ \rho_0 h_N(p_N) \end{bmatrix}}_{\mathbf{u}} \quad (64)$$

with the state vector $\mathbf{x} = [q_1, p_1, q_2, \dots, p_{N-1}, q_N]^T$, the adapted pressure differences $\widetilde{\Delta p}_i = \rho_0(h_{i-1} - h_i)$ with $i = 1, \dots, N$, the compression ratios $\chi_j = e^{\int_{p_0}^{p_j} \kappa(\tilde{p}) d\tilde{p}}$ with $j = 1, \dots, N-1$ and $N > 1$ as the number of pipe elements. The components of the input vector $\mathbf{u}$ represent the adapted pressures $\tilde{p}_0$ and $\tilde{p}_N$ at both ends of the pipe and, thus, the collocated output containing the corresponding flow rates reads

$$\mathbf{y} = \mathbf{g}_{\text{RLC}}^T(\mathbf{x}) \left(\frac{\partial E_{\text{RLC}}}{\partial \mathbf{x}} \right)^T = \begin{bmatrix} q_1 \\ q_N \end{bmatrix}. \quad (65)$$

Consequently, the energy stored in the pipeline calculates to

$$E_{\text{RLC}} = \sum_{i=1}^N \frac{1}{2} L_i q_i^2 + \sum_{j=1}^{N-1} V_j (\rho_j h_j - p_j) \quad (66)$$

and the Port-Hamiltonian formulation in PCHD structure follows to

$$\mathbf{J}_{\widetilde{\text{RLC}}}(\mathbf{x}) = \begin{bmatrix} 0 & -\frac{\beta_1}{\chi_1 V_1 L_1} & 0 & 0 & 0 & 0 \\ \frac{\beta_1}{\chi_1 V_1 L_1} & 0 & -\frac{\beta_1}{\chi_1 V_1 L_2} & 0 & 0 & 0 \\ 0 & \frac{\beta_1}{\chi_1 V_1 L_2} & 0 & -\frac{\beta_2}{\chi_2 V_2 L_2} & 0 & 0 \\ 0 & 0 & \frac{\beta_2}{\chi_2 V_2 L_2} & 0 & \ddots & 0 \\ 0 & 0 & 0 & \ddots & 0 & -\frac{\beta_{N-1}}{\chi_{N-1} V_{N-1} L_N} \\ 0 & 0 & 0 & 0 & \frac{\beta_{N-1}}{\chi_{N-1} V_{N-1} L_N} & 0 \end{bmatrix} \quad (67)$$

and

$$\mathbf{R}_{\widetilde{\text{RLC}}}(\mathbf{x}) = \begin{bmatrix} \frac{R_1}{L_1^2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{R_2}{L_2^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{R_N}{L_N^2} \end{bmatrix}, \quad (68)$$

which yields the expected change of the Hamiltonian along solutions of system (64)

$$\frac{d}{dt} E_{\widetilde{\text{RLC}}} = \frac{\partial E_{\widetilde{\text{RLC}}}}{\partial \mathbf{x}} (\mathbf{f}_{\widetilde{\text{RLC}}}(\mathbf{x}) + \mathbf{g}_{\widetilde{\text{RLC}}}(\mathbf{x}) \mathbf{u}) = \mathbf{y}^T \mathbf{u} - \sum_{i=1}^N R_i q_i^2 \leq \mathbf{y}^T \mathbf{u}. \quad (69)$$

So far, in the inductance model only constant lumped parameters L_i and R_i with a constant density ρ_0 were used, which is a good approximation for pressure levels higher than reference pressure p_0 , where the density changes only marginal with respect to pressure. For simulations in the low pressure range also low values of the density must be expected, which in turn have direct influence on the parameters L_i and R_i . For this reason, a pressure dependent lumped inductance can be defined by

$$L_i(p_{i-1}, p_i) = \frac{l_p}{r_p^2 \pi} \tilde{\rho}(p_{i-1}, p_i) \quad (70)$$

with a mass specific distribution of the density

$$\tilde{\rho}(p_{i-1}, p_i) = \frac{\rho_{i-1} V_{i-1} + \rho_i V_i}{V_{i-1} + V_i} = \rho_0 \frac{\chi_{i-1} V_{i-1} + \chi_i V_i}{V_{i-1} + V_i} \quad (71)$$

regarding the volumes V_{i-1} and V_i connected to the inductance element at both ends. It is remarkable that with this definition of a pressure dependent lumped inductance also the result from Equation (69) is preserved due to

$$\frac{1}{V_{i-1} \kappa_{i-1} \chi_{i-1}} \left(\frac{\partial E_{\widetilde{\text{RLC}}}}{\partial p_{i-1}} + \frac{1}{2} \frac{\partial L_i}{\partial p_{i-1}} q_i^2 \right) - \frac{1}{V_i \kappa_i \chi_i} \left(\frac{\partial E_{\widetilde{\text{RLC}}}}{\partial p_i} + \frac{1}{2} \frac{\partial L_i}{\partial p_i} q_i^2 \right) = 0 \quad (72)$$

for $i = 1, \dots, N$ with

$$\frac{\partial L_i}{\partial p_j} = \rho_0 \frac{l}{A} \frac{\chi_j \kappa_j V_j}{V_{i-1} + V_i}$$

and $j = i - 1, i$. It must be mentioned that in this modeling approach also the static resistance R_i is pressure dependent due to

$$R_i = \frac{8 \nu l_p}{r_p^4 \pi} \left(\frac{\rho_{i-1} V_{i-1} + \rho_i V_i}{V_{i-1} + V_i} \right) = \frac{8 \rho_0 \nu l_p}{r_p^4 \pi} \left(\frac{\chi_{i-1} V_{i-1} + \chi_i V_i}{V_{i-1} + V_i} \right). \quad (73)$$

Furthermore, at both pipe ends the volumes V_0 and V_N appear as new input variables of the pipeline model. For instance, in low pressure simulations of the damper with inductance model from Equation (50) the density must be calculated with regard to the chamber volumes of the cylinder Ax and $A(l - x)$.

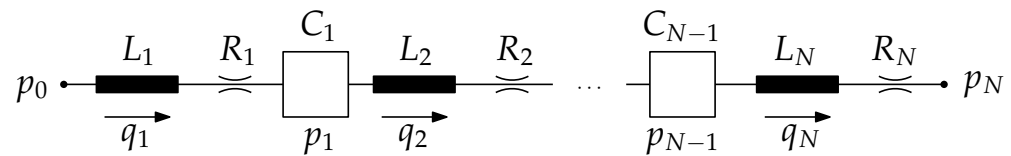


Figure 10. A pipeline approximated by RLC-elements with pressure boundaries.

In the following the simulation results for different RLC models for a pipeline with $l_p = 40$ m in length and a diameter $r_p = 5$ mm discretized by $N = 20$ RLC-elements according to a rectangular shaped input pressure p_0 and a constant pressure p_N at the opposite end are illustrated. The inlet pressure magnitude was selected to ensure laminar flow conditions within the pipeline. In the following the simulation results for three different RLC models parameterized with regard to Table 2 are presented; the $\widetilde{RLC}$ model incorporating mass flow compensation with the compression ratios χ_i and maintaining the energy balance with $\widetilde{\Delta p_i} = \rho_0(h_{i-1} - h_i)$ is opposed either to a simple $\overline{RLC}$ model without any corrections ($\chi_i = 1$, $\widetilde{\Delta p_i} = p_{i-1} - p_i$) or to an $\widehat{RLC}$ model with mass flow compensation χ_i , but where energy is not preserved in the inductance elements ($\widetilde{\Delta p_i} = p_{i-1} - p_i$). Furthermore, in the $\overline{RLC}$ and $\widehat{RLC}$ models constant parameters L_i and R_i are used, while the $\widetilde{RLC}$ model considers a pressure dependent density in the corresponding parameters. In the upper diagrams selected pressure states p_i with $i \in \{2, 7, 12, 17\}$ of the respective capacitances at equally spaced locations along the specific pipe models and the input pressures p_0 and p_{20} are shown. In the middle diagrams the flow rates of the first inductances q_1 are opposed. In the lower diagrams the difference between the illustrated flow rate results on port one is depicted.

Table 2. Different configurations of the RLC model from Equation (64).

Model	Property	Characteristics
$\overline{RLC}$	linear ($\varphi_0 = 0$)/non-linear ($\varphi_0 > 0$)	$\chi_i = 1$, $\widetilde{\Delta p_i} = p_{i-1} - p_i$
$\widehat{RLC}$	non-linear	$\chi_i = \frac{\rho_i}{\rho_0}$, $\widetilde{\Delta p_i} = p_{i-1} - p_i$
$\widetilde{RLC}$	non-linear	$\chi_i = \frac{\rho_i}{\rho_0}$, $\widetilde{\Delta p_i} = \rho_0(h_{i-1} - h_i)$

In Figure 11 the fluid model from Equation (4) without entrained air ($\varphi_0 = 0$) was used with a constant bulk modulus β . The simulations show a noteworthy difference between both models at moderate pressure levels. In contrast to that in Figure 12 the results for an advanced fluid model with entrained air regarding Figure 4a at low pressure levels are presented, where the differences between the models are significantly lower than at higher pressure levels. A possible reason seems to be that the density and the enthalpy are monotonically increasing with pressure. Furthermore, it is remarkable that despite of the constant lumped parameters L_h and R_h in the models $\overline{RLC}$ and $\widehat{RLC}$ the errors compared to the $\widetilde{RLC}$ with pressure dependent parameters are quite low. Probably, the error due to the defects in mass flow and energy balance is almost compensated by the effect of constant lumped parameters compared to physically correct modeling at such low pressure levels.

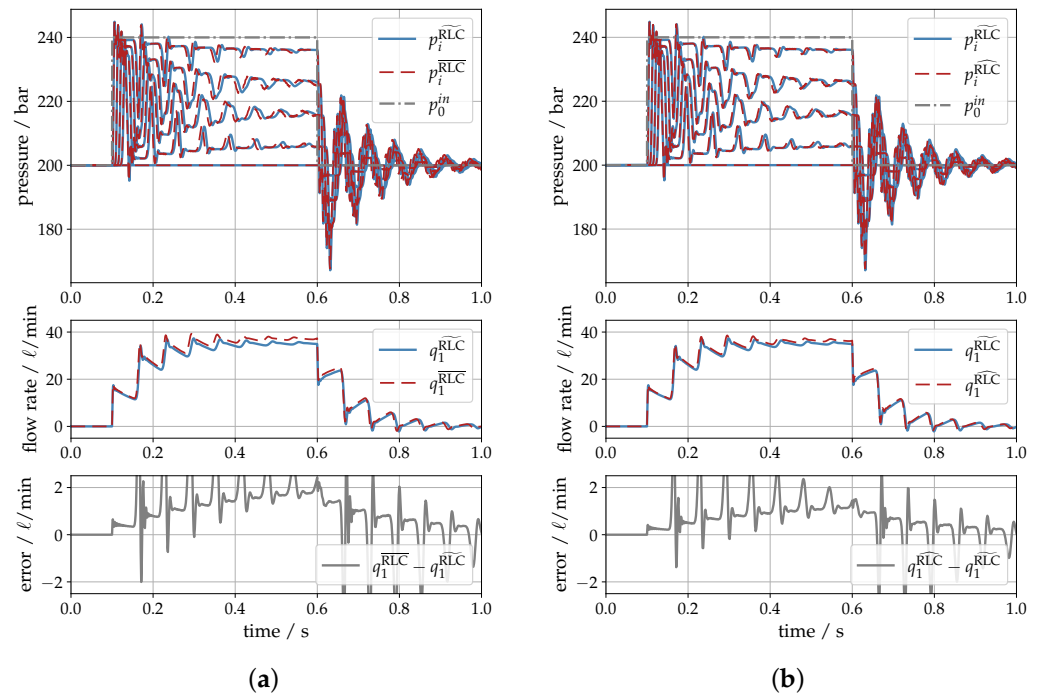


Figure 11. Simulation results for the pipe models with the simple fluid model from Equation (4). (a) RLC vs. RLC without mass flow compensation; (b) RLC vs. RLC with mass flow compensation.

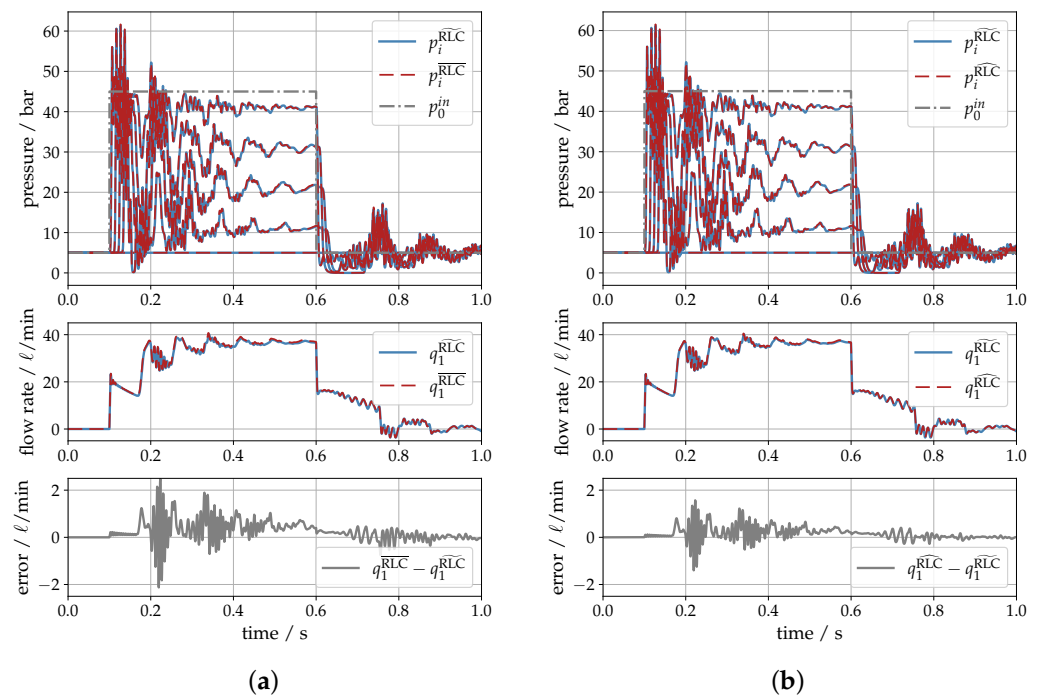


Figure 12. Comparison of pipe models, all with advanced fluid model from Equation (46) at low pressure levels. (a) RLC vs. RLC without mass flow compensation; (b) RLC vs. RLC with mass flow compensation.

5. Discussion

5.1. Passivity

If the change of the energy stored in a system is lower or at most equal to the power supplied to the system through its ports, then the system is called to be passive. According to this definition the power flow into or within a system is the key quantity for an evaluation of a system with regard to meaningful physical behaviour.

In hydraulic engineering it is often suitable to quantify the hydraulic power by the product of volumetric flow rate and pressure difference, because both are measurable quantities. But from the investigations above it became clear that this is just an approximation of the actual power. Because during the flow through models based on an incompressible approach the mass balance is violated to some extent. Most often the corresponding error is negligible, but in a rare number of applications, like in closed systems as presented above with considerably low amount of oil, the unbalanced mass flow results in totally meaningless simulation results. For example, in contrast to the approximated power by pressure and flow rate, the power dissipated in a hydraulic resistance is correctly represented by the mass flow multiplied by the enthalpy difference of the fluid between both ports.

5.2. Compression Ratio of the Fluid Density

One important finding of this contribution is that with the definition of the compression ratio the mass balance in a hydraulic system consisting of lumped parameter models can be fulfilled without the necessity of a model representation in canonical coordinates and thus, the representation in sensor coordinates can be maintained. This is particularly advantageous in simulations, where the models are sensitive against an unbalanced mass flow. In most cases, however, depending on the fluid model, the mass balance compensation can be achieved just by adding simple function blocks to the already existing overall simulation model; no additional library or toolbox is necessary. For the application of more sophisticated fluid models, where no analytical representation of the compression ratio can be achieved, an increased computational effort must be expected, since an integral has to be numerically evaluated multiple times by the solver for each time step.

The simulations with a mass flow compensation using the compression ratio showed even satisfying results in the low pressure range for a fluid model with a pressure dependent constitutive law of material. It must be remarked that in this study oil hydraulic drive systems were investigated, where the amount of entrained air is quite low.

5.3. Pipeline Models

The lumped modeling of transmission lines with RLC elements is a widely applied method in hydraulic engineering, because at moderate pressure rates the pipeline dynamics can be formulated as a linear ODE system, if a fluid model without entrained air and, thus, a constant effective bulk modulus is used. However, as the investigations from above revealed, in the linear RLC model the mass balance and the energy balance are violated. On the one hand the unbalanced mass flow results in a slight loss of fluid mass over the lumped resistances in the inductance, which in turn leads to a higher mean flow rate through the pipe, since lower inertia is more accelerated by the same input pressure. Furthermore, in the linear model the flow rate through the inductance elements is even more increased by the unbalanced energy, because in this case the pressure difference over each pipe element is solely used for the acceleration of the fluid. In contrast to that in the nonlinear model among the fluid acceleration the input pressure is also used for the compression of the fluid in the enthalpy.

At low pressures also the simple unbalanced RLC model is nonlinear, because in this pressure range the fluid model with entrained air must be used in order to prevent negative pressure values in the capacitances. Furthermore, the simple RLC model uses constant lumped parameters in the ODEs for the inductance elements, while the parameters of the capacitance elements were pressure dependent. But this effect is only relevant at pressures below reference pressure. It is more remarkable that at pressures slightly above the reference pressure the error between unbalanced and balanced modeling is significantly lower than in the high pressure range. This can be explained by the fact that the enthalpy

and, thus, the internal energy in the fluid is lower, which reduces the error compared to the simple model, where this effect is not considered.

In the pipe models investigated in this contribution only a static laminar resistance was considered, which is actually only in a steady state laminar flow a good representation of reality. In order to achieve a more realistic approximation of dynamic dissipative effects in transmission lines frequency dependent friction must be taken into account. In [34] a possibility for the modeling of frequency dependent friction in combination with lumped parameter RLC pipe models is presented. But, if the frequency dependent friction model fulfills the passivity condition within this modeling approach is not investigated so far, which could be next steps for future investigations.

5.4. Simulation Results

The presented investigations reflect the different modeling approaches based on simulation experiments, which in fact are different approximations of reality, but the results are not yet confirmed by corresponding measurements. However, the presented strategy in lumped parameter hydraulic modeling leads at least to a meaningful physical interpretation with regard to Hamiltonian systems theory and the passivity of the corrected models could be shown, which is an important property for the interconnection with other physical models. Nevertheless, a validation of the simulation experiments with measurements particularly for the closed hydraulic damper systems will be part of future work.

6. Conclusions

The dynamic modeling of hydraulic drive systems with lumped parameter models requires a correct consideration of the relevant physical effects in the corresponding domains as mechanics, fluid dynamics and thermodynamics. In this paper the deficiencies of conventional models from literature were identified by simulations and by using the framework of Port-Hamiltonian systems. The investigations showed that the modeling of dynamic hydraulic effects in sensor coordinates is in fact a simple and intuitive approach for hydraulic engineers, but results to some extent in a violation of mass- and energy-balance, which in turn leads to a loss in accuracy in certain simulation cases. It was shown that the mentioned shortcomings can be corrected without losing the convenient system representation in sensor coordinates by only minor extensions of the model structure, which can be easily realized in existing simulation tools. Furthermore, the modeling of wave propagation effects in a transmission line approximated by lumped RLC elements was investigated with the PCHD framework. With the proposed modifications to the model formulation, the passivity of all investigated lumped models could be successfully demonstrated. The results contribute to a systematic framework for the physically consistent modeling and interconnection of multi-domain systems.

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Abbreviations

The following abbreviations are used in this manuscript:

BDF	Implicit multi-step variable-order (1 to 5) method
CFD	Computational Fluid Dynamics
DAE	Differential-Algebraic Equation
DOP853	Explicit Runge-Kutta method of order 8
LSODA	Adams/BDF method with automatic stiffness detection and switching
ODE	Ordinary Differential Equation
PCHD	Port Controlled Hamiltonian System with Dissipation
PDE	Partial Differential Equation
PWM	Pulse Width Modulation
Radau	Implicit Runge-Kutta method of the Radau IIA family of order 5
RK23	Explicit Runge-Kutta method of order 3(2)
RK45	Explicit Runge-Kutta method of order 5(4)
RLC	Resistance (R), Inductance (L) and Capacitance (C)

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